Natural Hazard Research

PROBABILISTIC APPROACHES TO DISCRETE NATURAL EVENTS: A REVIEW AND THEORETICAL DISCUSSION

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PREFACE

This paper is one in a series on research in progress in the field of human adjustments to natural hazards. It is intended that these papers will be used as working documents by the group of scholar directly involved in hazard research as well as inform a larger circle of interested persons. The series is now being supported from funds granted by the U. S. National Science Foundation to the University of Chicago and Clark University. Authorship of papers is not necessarily confined to those working at these institutions.

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PROBABILISTIC APPROACHES TO DISCRETE NATURAL EVENTS:
A REVIEW AND THEORETICAL DISCUSSION

1. Introduction

It is frequently necessary or useful to examine the probabilistic structure of events comprising only a fraction of a time or space continuum. This paper is concerned with such situations where they involve natural events or their impact on human communities. The subject-matter range from rainy periods, dates of plant germination, and sea-waves, to landslides and tornado strikes. We shall deal largely with statistical models relevant to groups or series of events of a single type, both in terms of counts of undifferentiated instances and measures of magnitude of event. Interest will be centered on probabilistic models, their physical interpretations, and logical problems of equivalence relations with natural phenomena. No attention will be given to the techniques of analysis, or curve-fitting.

In assessing the expectancy of an event from a series of measurements of like events, the possibilities to be considered are:-

i) That the probability distribution arises from the independence and random occurrence of the events.

ii) That the distribution arises for independent events with an imposed pattern in space or time (e.g. Renewal processes).

iii) That the distribution arises from random events but with dependence between individual events (e.g. "after-effects," "contagion").

iv) That the probability of an event at any time or place is partly or wholly controlled by a "deterministic" process as with periodicity, or trend.

v) That the series of events derives from more than one process and represents the mixing or pooling of more than one probability function.

These are certainly not mutually exclusive possibilities, but they indicate important logical differences which help sub-divide material in a paper of the present type. We shall concentrate on i), ii) and iii) only.

The basic observations are counts of events, and measures of the intervals in space and/or time between events. Where these are the only observations we deal with "counting processes" in the wide sense. If, in addition, there are measures of event-variables to take account of - intensity, duration, size -, we have a magnitude-frequency problem. The magnitude scale need not be discrete. In many cases, discrete phenomena may be treated as point processes, but considerations of magnitude can involve duration or areal extent as variables. A further common distinction is made between frequent and "rare" events, and where magnitude is involved between "average" and "extreme" events. While highly relative in a time or space sense, these distinctions have a basis in practical and conceptual differences, even when the infrequent is interpreted with special or limiting cases of distributions suited to frequent events.

1.2 The Definition and Identification of Discrete Events

The title of the paper implies the popular rather than statistical use of the word "event." That is, we are concerned first with recognisable occurrences in nature. Equivalence to statistically defined "events" must be determined for given instances. A discrete natural event will be treated as an occurrence with instanteous or limited duration in a time continuum. While spatial probabilities of time-limited events will be considered, it is the limited duration which defines the interest. Discrete event problems arise in two different and logically independent situations; either when observations are made at discrete intervals giving "operationally" discrete events; or where the event is bounded in time by physical criteria which apply even with continuous observing.

Discrete events may be identified by one or more of the following criteria:-

i) The "self-evident" phenomenon; an unanalysed package which a large proportion of the observing community feel they know when they see it. Blizzard, tornado or comet records include many with only a visual report.

ii) Threshold events, in which the event occurs when a given value on a measurement scale is reached, crossed or surpassed. The threshold may, or may not have a clear physical meaning. A special case is where all non-zero measurements constitute events or the binary situation where an event is analogous to the "on" position of a switch.

iii) "Coincidence," in which the event requires two or more conditions to occur together, or when thresholds for two or more variables are equalled or exceeded jointly. A special case is involved in the study of "singularities" or "spells," - the joint occurrence of different events or of certain events on particular dates (Shapiro and MacDonald, 1961).

All three criteria may apply either to conventional measurement events or to physically structured events. Either way they are the means of deciding when an event occurs. If the criteria work unambiguously, and we are only interested in counts of events then one may proceed directly to statistical considerations. However, if the aim is anything but exploring the data, this approach is inadequate. Unless there are physical postulates, or models identified with measurement then it is hard to say what the meaning of the most refined analysis of a string of measurements is. It is fair to say that in much of the material of interest to us, such matters have received scant attention. The case is exemplified by the many analyses of, say, river level maxima or minima and the negligible work on the frequency distribution of flood-waves which are more-or-less well-defined physical entities related to specific controlling conditions. But we will return to these issues in discussions, and especially in the last section.

2. A Survey of Probability Models for Discrete Natural Events.

2.1 Introductory Remarks

The value of probability theory to modern science was first demonstrated by those who used it in the basic physical postulates from which they derived or replicated observed behaviour; and with results more convincing than deterministic theories. Geographers are returning to this use of probability theory. While much of the relevant literature on discrete phenomena still emphasises problems of sampling, statistical estimation and empirical curve-fitting, there is an awareness that if all this is simply to overcome data volume or to reach "deterministic" statements with an estimate of error, it is a pretty crude sort of science.* The situation is aggravated by the high order events that interest us, - harvest yields, floods, accidents, - which encourage the use of statistical methods to identify relative degrees of control by the many variables we feel are at work, rather than enquiry into the probabilistic nature of the underlying processes.

The present paper is directed towards probabilistic questions and will examine statistical distributions primarily in terms of how and under what conditions they can arise. A range of natural phenomena which have been analysed and fitted more-or-less adequately to various distributions is given in Table I at the end. We begin by introducing most of the relevant distributions in terms of undifferentiated events, although all of these distributions can also be used to characterise aspects of magnitude of event.

2.2 Frequency and recurrence intervals of independent random events

In all the cases considered in this section, the events themselves are assumed to satisfy the relation:-

$$p(n_i) = p(n_i + I) \quad \text{for all } n_i \tag{1}$$

*Some would protest this. Karl Peason, no less, condemned Kapteyn's use of the distribution generating approach, as opposed to his own curve-fitting (c.f. Aitchison and Brown, 1957).

Let us begin with a simple, hypothetical case analogous to common substantive situations, and build upon it. Suppose that a certain lake is visited once a week for checking and we need to know how often lake-level will be within certain limits. The data consist of a long record of once-weekly readings. For an infinite population of readings the relative frequency $\phi(x)$, of encountering the lake within limits $x \pm\Delta x/2$ is found from:-

$$P = \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \phi(x)dx \qquad (2)$$

We have discrete data and are only interested in two states: that the lake is within limits (p), or not within limits (q), so that:-

$$q = 1 - p \qquad (3)$$

If the mass-balance of the lake were a random variable, and readings a week apart independent, we have an exact analogue of the data in Bernoulli trials using an urn with p and q the same. If the levels of interest are close to mean lake level and the limits moderately broad p should be a large fraction. The number of occasions k when we expect to find the lake in limits out of n visits can be found from:-

$$p(k;n,p) = \binom{n}{p} p^k q^{n-k} \;;\; k = 0,1,2,\ldots \;;\; 0 \le p \le 1 \qquad (4)$$
$$= 0 \qquad \text{otherwise}$$

THE BINOMIAL DISTRIBUTION

There is no essential difference between the frequency distribution of our data and of the urn model. Had we, alternatively, been interested in the events net rise (p_1), net fall (p_2) or no change (p_3) between successive readings, with these again independent random variables we have a good approximation in:-

$$p(k_i;n,p_i) = \frac{n!}{k_1!\,k_2!\,k_3!}\,p_1^{k_1}p_2^{k_2}p_3^{k_3}$$

and generally:-

$$p(k_i;n,p_i) = \frac{n!}{k_1!\,k_2!\,\dots k_r!}\,p_1^{k_1}p_2^{k_2}\dots p_r^{k_r};\quad k_1,k_2,\dots k_r = 0,1,2,3,\dots,\quad k_1+k_2+\dots k_r = n;\quad p_1^{k_1}+p_2^{k_2}+\dots p_r^{k_r} = 1 \qquad (5)$$

THE MULTINOMIAL DISTRIBUTION

Again, the distribution can be derived from an appropriately structured urn model (c.f. 2.4.). The Binomial is obviously a particular case of the Multinomial, but much more widely used.

Natural situations are rarely, if ever as simple as our example. Furthermore, one is led to question the meaning of treating discrete readings as physical "events." It is our object to examine such questions, and we will begin by attempting to define the kinds of physical conditions under which a symmetrical Binomial distribution can arise in nature. Griffiths (1967, p.256) lists the following:- Quote:

1) In the absence of certain specific "causes" of variation, or in the event of perfect balancing of these effects, the data may assume a central fixed value.

2) Deviations from the central value result from certain causes of variation, the effect of any cause being either to add a fixed quantity or subtract the same quantity.

3) The probability that a cause of variation will produce a positive effect equals the probability that it will produce a negative effect:

$$P(+ve) = P(-ve) = \frac{1}{2}$$

4) The net effects of all contributing causes of variation are of equal magnitude in either direction.

5) The contributory causes are independent in their action; the P(+ve) or the P(-ve) from any causal factor is independent of previous contributions.

6) The total deviation of any element from its central value is the algebraic sum of positive and negative contributions of the individual causal factors.

I would want to interpret his "causes" in a wide sense to include possible micro-states, interactions, responses, stresses and any other physically defined relation or contact.

A case in which spatial occurrence should approximate a Binomial distribution would be the frequency of raindrop impacts per unit area of a level field or pond during a steady rainstorm. Likewise, the wavelets spreading out from each impact point on the pond would be expected to have randomly distributed phase angles. A count of frequency of ripple crests at a point would then approximate the Binomial distribution. This same reasoning has been applied with moderate success to the statistical theory of wave generation in the open ocean, where the wind waves are assumed to arise from random sources (c.f. Longuet-Higgins, 1950; Pierson, 1954). However, here as elsewhere the preference for employing the continuous Normal or Guassian distributions prevails.

The importance of the Binomial distribution and its discrete relatives lies mainly in their heuristic value for developing, say, equilibrium theories of random situations and examining how deviations might arise from these states. From simple, easily understandable postulates one rapidly derives "realistic" and complex conditions. An elegant example is found in Schrödinger (1946), while Feller (1957), develops examples using urn models. In particular, the analytically more powerful Normal distribution, which is central to modern statistics, can be built up using the logic of the Binomial model. As is proved by the Laplace-de Moivre Limit Theorem (e.g. Sokolnikoff and Redheffer, 1966, p.623), as $n \to \infty$ the binomial variable k in Equation 4 is asymptotically normal, with mean $np = m$, and variance $np(1-p) = \sigma^2$. The well-known density function is:-

$$p(k;m,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\frac{k-m}{\sigma}\right)^2\right] \; ; \; -\infty<m<\infty; \quad \sigma > 0 \qquad (7)$$

THE NORMAL or GAUSSIAN DISTRIBUTION

Use of this distribution is prevalent throughout most of the work of interest. The conditions under which a physical process follows a Normal frequency function are related to those for the symmetrical Binomial. It is further required that the number of contributary "causes" be large, and their positive and negative contributions small (c.f. 2.4. p.). In that case only Griffiths' point 6) need be fully satisfied and 4) approximately. For large samples, our rain splashes, the distance between them, and the phase angles of resulting waves should follow a Normal distribution. However, as in Cartwright's work on sea-wave variables (see Table I) and most work in hydrology, the assumption of normality may be adopted as a convenient rather than strictly accurate feature (Cartwright, 1962). A further property of this distribution is the relative ease with which many non-normal distributions may be transferred to, or approximated by the Normal (c.f. 2.4. p.). In addition to the Binomial and Hypergeometric, Govindarajulu (1965) has derived uniform normal approximations to the discrete distributions, Poisson and Negative Binomial, to which we now turn.

Far more common than the Binomial in the study of discrete random events is the Poisson distribution. This is a discrete distribution associated with fairly infrequent events; say, where $p < 0.2$. It is easily shown how the basic Poisson distribution will arise from the urn model for the Binomial but with $p \ll 1$ and $k \ll n$; the "rare" event model. Let a parameter λ be the mean sample count (of "successes") for an infinite number of samples, - the statistical <u>expectation</u> of k, - and then write the Binomial distribution as:

$$p(k) = \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

Let n increase towards infinity while λ decreases but remains greater than zero, and:

$$p(k) = \frac{1}{k!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} [n(n-1)\ldots(n-k+1)]$$

$$\simeq \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k} \left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \ldots, \left(1-\frac{k-1}{n}\right)\right]$$

If n increases towards infinity while λ and k remain finite, the term in braces converges to a value of I, while the middle term will give:

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n-k} \simeq \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n}$$

$$= \lim_{n\to\infty} \left(1 - \lambda + \frac{n(n-1)}{2!n^2}\lambda^2 - \ldots\right)$$

$$= 1 - \lambda + \frac{\lambda^2}{2!} - \ldots)$$

$$= e^{-\lambda}$$

Thus we arrive at the Binomial approximation to a distribution given by:

$$p(k;\lambda) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad ;k = 0,1,2,\ldots;\lambda > 0 \qquad (8)$$

$$= 0 \qquad \text{otherwise}$$

THE POISSON DISTRIBUTION

As with the Binomial we can go on to define:-

$$p(k_1,k_2,\ldots,k_n) = e^{-(\lambda_1+\lambda_2+\ldots\lambda_n)}\frac{\lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_n^{k_n}}{k_1!\,k_2!\,\ldots k_n!} \qquad ;k_1,k_2,\ldots,k_n = 0,1,2,\ldots;$$

$$\lambda_m > 0, m = 1,2,\ldots,n.$$

$$= 0 \qquad \text{otherwise} \qquad (9)$$

THE MULTIPLE POISSON DISTRIBUTION

But I have not found a substantive use of this in our field.

In analyses of natural events, the Poisson is the "bad news" distribution associated with such phenomena as waterspouts, floods, severe storms, meteorite strikes and, most famously, horse kicks in the Prussian cavalry. Again, however, few natural phenomena give more than a rough approximation to the distribution, and even with a close fit, to demonstrate that a substantive distribution arises from a <u>Poisson process</u>, such as our modified urn model, requires them to fulfil fairly stringent criteria. These criteria are:-

i) The probability of more than one event in a small interval of length y is of a smaller order of magnitude than y, such that the probability is $o(y)$ as $y \to 0$. (i.e. the chances of more than one event occurring simultaneously or at the same spot are negligible).

ii) That the probability $p_n(y)$ of the occurrence of a specified number n of events in a given interval y, depends upon n and y but is independent of where the interval is chosen in the overall space.

iii) The number of events in different segments of the sample domain represent independent random variables. (i.e. a time or space series of discrete points must be stationary).

When referring to an appropriate process we use a distinct notation which in the case of a Poisson process might be:-

$$P\{X(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad ; \text{ etc.,} \qquad (8a)$$

or;

$$P\{X(t) = k\} = e^{-A} \frac{(\lambda A)^k}{k!} \quad ; \text{ etc.,} \qquad (8b)$$

where t and A are index parameters consisting of real, positive numbers respectively of time and area, and form terms for the time or space averages (for detailed discussions see Cox and Lewis 1966; Purzen, 1962). In general, it seems that the Poisson fits natural events best where λt or λA are very small compared to the sample period or quadrat (e.g. Thom 1957, who found a good fit for hail storms only with $\lambda<2$ per year). Otherwise, "clustering" and at least the appearance of dependence between events tend to occur. The criteria thus violated can, however be circumvented using modified Poisson or other models (see 2.3).

Some of the most important distributions used in studying discrete natural events can be developed from those already discussed if we consider the recurrence intervals of, or waiting time between events. There are two questions which can be asked: i) how long, or how many trials must one have to record a single event ("success")? and more generally, ii) how long, or how many trials are needed to record r events? Answers to these questions involve distributions with positive skewness. The discrete distribution which answers the first question for a series of Bernoulli trials has the probability mass function:-

$$p(x) = p(1-p)^x \; ; \; x = 0,1,2,\ldots; \; 0 \leq p \leq 1 \qquad (10)$$
$$= 0 \qquad \text{otherwise}$$

THE GEOMETRIC DISTRIBUTION

It is a much less common distribution than the continuous equivalent applying to the same question. Given that the events are occurring randomly at a mean rate λ per unit of time we have:-

$$p(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$
$$= 0 \qquad \text{otherwise} \qquad (11)$$

THE EXPONENTIAL DISTRIBUTION

This is a fairly versatile distribution and may adopt a number of forms (e.g. 2.4.).

In answer to the second question; the discrete distribution which gives the number of trials up to the rth "success" in Bernoulli trials is:-

$$p(x) = \binom{r+x-1}{x} p^r q^x$$
$$= \binom{-r}{x} p^r (-q)^x \quad ; x = 0,1,2,\ldots; \; 0 \leq p \leq 1$$
$$= 0 \qquad \text{otherwise} \qquad (12)$$

THE NEGATIVE BINOMIAL or PASCAL'S DISTRIBUTION

This is actually called Negative Binomial where r is an arbitrary real number; and Pascal with r a positive integer. With r = I it reduces to the Geometric, and is also a limiting form of the Polya distribution (see Feller op. cit. p. 131).

To answer the second question with a continuous distribution when events are occurring at a mean rate λ we may use:-

$$p(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} \quad ; x > 0$$
$$= 0 \qquad ; x \leq 0 \qquad (13)$$

THE GAMMA or PEARSON TYPE III DISTRIBUTION

Where $\Gamma(r)$ is the incomplete Gamma or "factorial" function, i.e. $\Gamma(z)$ with $z = r$, a positive integer such that $\Gamma(r) = (r-1)!$, (see Sokolnikoff and Redheffer op. cit. p. 45). This is a very versatile distribution. It reduces to the Exponential with r = 1. The Gamma and Exponential distributions are used extensively to describe recurrence intervals or waiting-times in a wide variety of situations, especially with Poisson and related processes (e.g. Parzen op. cit.). In a recent paper careful comparisons were made between these two distributions and a Markov model (see below) to describe waiting-times between river level exceedances for modified daily series', (McGilchrist et.al. 1968 and 1969).

Decker (1952) used a Gamma model to fit the spatial occurrence of damaging hail storms.

We have derived these models in a particular way. They may however be reached through other processes. The Normal,Exponential and Gamma laws are most often applied to magnitude-frequency problems, while the Negative Binomial and Gamma are important in the study of "after-effects" or clustering in discrete events, to which we now turn.

2.3 Clustering, After-effects and Contagion in discrete events

In fact, the most erratic of natural events have a tendency to occur in clusters even when not confined to particular times. Tornadoes, seismic tremors,sunspots and meteorites are recorded in swarms. With discrete observing such phenomena as high or low river flow, rainy or dry days, tend to occur in "runs." The physical interpretation of apparent dependence is difficult, but first we look at the kinds of probabilistic processes which result in such distributions.

The simplest type of dependence between random events is where a state or event depends upon the immediately preceding event but on no other. Physically, this might arise where there are chance encounters but specific outcomes according to type of encounter; or where conditions with a certain probability structure "trigger" others differently structured or, most commonly, where there is physical continuity in a phenomenon moving randomly in time or space. The process is analogous to an urn model with several urns of differing composition, each trial prescribing from which urn the next trial is taken (see Feller op. cit., p. 339). In such a case we have a discrete "Markov chain" in which, for every finite set $t_1 < t_2 < \dots\ t_{n-1} < t_n$:-

$$P\{(x_n, t_n | X_1, t_1 \ ; \ \dots \ ; \ X_{n-1}, t_{n-1})\} \\ = P\{(X_n, t_n | X_{n-1}, t_{n-1})\} \tag{14}$$

THE SIMPLE MARKOV PROCESS

A process of this type is fully specified by its first-order probability distribution together with the "transitional probabilities":-

$$P(X_2, t_2 | x, t)$$

Such models have been applied to hydrologic and climatic time-series with moderate success, but usually after considerable modification of the original observations (e.g. Gabriel and Neumann, 1962, McGilchrist et.al. 1968; Caskey 1964; Weiss, 1964).

Now, consider an important distinguishing feature of the Negative Binomial distribution, which is its moment measure:-

$$\frac{\text{variance}}{\text{mean}} > 1$$

whereas $\sigma^2/\lambda < 1$ for the Positive Binomial and $\sigma^2/\lambda = 1$ for Poisson. One way in which a large variance can arise in nature is through clustering of events at particular times or places, so that the mean is an unstable estimate of the number of events in an arbitrary unit. The Negative Binomial, and for similar reasons the Gamma, may be an excellent fit for these situations. But now, the data do not exhibit the statistical independence under which we first developed these models. Clearly, if the distributions are to arise from a single process, then somehow some "successes" increase the chances of recording more successes. For instance, in our urn model we would need an arrangement whereby, say, a success from our original urn would direct sampling to another urn with larger p, or lead us to add "success" units to the urn, (Feller, op. cit. pp. 109-114; and Dacey 1964, p. 961 give exact specifications for urn schemes with conditional probabilities). Only fairly simple urn and card-pack models of clustering in nature have been satisfactorily developed.

To take account of persistence in series of events occurring roughly as a Poisson process, Eggenberger and Polya (quoted in Brooks and Carruthers, 1953) developed a model in which the terms of the standard Poisson series expansion are replaced by a "persistence" series (Wahrscheinlichkeitsansteckung):-

$$\frac{1}{(1+\beta)^{\lambda/\beta}}, \quad \frac{\lambda}{1!\,(1+\beta)^{\lambda/\beta+1}} \quad \frac{\lambda(\lambda+\beta)}{2!\,(1+\beta)^{\lambda/\beta+2}} \quad \ldots\ldots$$

with the general term:-

$$\frac{\lambda(\lambda+\beta)\ (\lambda+2\beta)\ \ldots\ldots[\lambda+(n-1)\beta]}{n!\,(1+\beta)^{\lambda/\beta+n}}$$

with λ the mean density per sample period (or quadrat) and β to be found from $\sigma^2/\lambda-1$. Brooks and Carruthers (op. cit.) find a good fit for observations of frost at Greenwich, England and quote analyses of thunder, gales, rain, cloudiness, fog, snowfall and snow-on-the-ground, which appear to follow this distribution (their Table CXII, p. 317). They suggest that the term $(1+\beta)$ can be called the "persistence factor" s and conclude that, "...to a first approximation a phenomenon having a persistence factor s has a tendency to occur in groups of average size s. This causes runs to be s-times as long and the variance of numbers of occurrences in a specified time to be s-times greater than would be expected if there were no peristance" (p. 318). It has been brought to my attention that Dacey (1965) gave an elaborate development of this model for spatial point processes. Although I have not seen it used it would seem highly relevant to such phenomena as tornado-strikes and hail-swaths.

The above distribution, and especially the Negative Binomial are two of a group of distributions associated with the idea of "contagion." Here, we adopt a strictly physical interpretation of "contagion"; namely, material interaction whereby one event directly increases the chances of another. In fact, almost any probability distribution can be modified to model contagion. The derivation and properties a various "contagious" distributions appear in Patil, 1964, Ch. 2 and 7, including a behavioural model of developing contagion in a Binomial situation. The kind of natural situation where the latter might apply would arise from our previous examples. With rain-splash erosion on a field during a steady downpour, the impact distribution remains Binomial, but each impact can modify the surface so that, looked at as erosional events, the distribution over some minutes of the storm can have both trend and contagion.

However, the physical interpretation of contagion is complex and open to many errors. Clustering of events may arise simply through mixing of random processes such that the events are independent but their distribution multimodal. Uneven observing can produce clustering, as well as material interaction. Even more difficult is the fact that many natural events are independent of one another but relate back complexly to common controls.

Some while ago, I became excited by the excellent fit of 150 years' record of natural dam-burst floods on the Upper Indus to a "contagious" Poisson of the Polya-Eggenberger form. It seemed to support the idea that even such extreme events were broadly related to certain clima-geomorphic controls. In fact it has proved impossible to determine whether the distribution is "real" and related to phases of climatic deterioration, or, among other things, purely the result of spatially and temporally uneven observing.

Summarising the general problem of contagion in biosciences Katti and Sly (1964) found that:-

Quote: "i) No single theoretical distribution has been found to describe any large scale data.

ii) For a number of data there could be two or more theoretical distributions that fit equally well and there is no way to choose between them based on fits alone.

iii) Two or more physical models could lead to the same final statistical distribution and hence the estimation of the parameters of the distribution may not have unique meaning.

iv) ... different methods of estimation lead to widely differing estimates when the methods are consistent... there are a number of empirical frequencies to which the same theoretical frequency function has been fitted by different consistent methods..."

Their response to this situation is in keeping with the approach of the present paper: namely, tie the statistical methods as closely as possible to physical or behaviouristic postulates.

An example which illustrates the above points is the use of the Negative Binomial. Various authors point out that a distribution fitting this function cannot be uniquely associated with a "negative binomial process" (Feller, 1943; Anscombe,1949; Bliss, 1953). For instance the clustering may arise, not through contagion but from heterogeneity. In particular, a process devised by Greenwood and Yule (1920) called an infinitely compounded Poisson model, and used for attacks of disease, gives identical distributions to the contagious Negative Binomial (Feller, op. cit. and also Harvey, 1968).

One final question to be noted is the physical meaning and value of persistence in discrete observation series! For inventory problems at a dam, say, there is a real difference between one-storm flood-waves lasting one day, and those lasting several days. However, apart from duration, what is the physical difference here that makes one "contagious" from daily readings, and the other random and independent? Any item can form "runs" if observing is sufficiently fine-grained. One major development which partly removes this problem is the use of autocorrelation and spectral analysis but these are outside our terms of reference here.

2.4 Magnitude and Frequency of Discrete Events

The probability of events of various magnitudes is of major concern in such fields as long-range forecasting, water resource development, and natural hazards, and has become a central issue in geomorphology (Wolman and Miller, 1960). Magnitude need not have the same statistical attribas undifferentiated counts even in the same event-type. One may obey an arithmetic, the other a logarithmic law. Magnitude can be an independent random variable while frequency has trend or clustering, and vice versa. For instance, sharply peaked flood waves may occur randomly throughout the year; but their at-a-station magnitude, which depends among other things on volume of water already in the channel, may be seasonally determined. Occasionally, magnitude is a discrete variable. Energy levels in the electron cloud of an atom, and wave-lengths of radiation emitted at changes of level are classic examples. The damaging impact of natural hazards may related significantly to attainment of certain thresholds, say, of areal extent or mechanical loading. Strength of building materials, or elevation of different forms of land-use on a flood plain will create such thresholds. In most cases, however, magnitude in natural phenomena is continuous over part or all of the range of interest.

A simple but powerful heuristic device for exploring probability structures here is some variant of "occupancy" models. For instance, take a set of boxes and stack them to form pigeon-holes. Let each column of boxes belong to one item, and each row represent a level of magnitude. It could be height, mass, areal extent; we will use "energy level." The single items in each column will be distinguishable but physically identical packages, and we can begin with the following:

i) $n_j \equiv$ number of packages on an energy level.

ii) $N = \sum n_j$; number of packages in the system.

iii) $u_i \equiv$ energy level (i.e. magnitude) of i^{th} row.

iv) ε_j $\equiv$ energy of the j^{th} package.

v) U $=\sum \epsilon j$; energy of the overall system.

vi) <u>With bounds</u>: max $\{\varepsilon j\} \leq U$; min $\{\varepsilon_j\} \geq 0$

vii) <u>And conditions</u>: $\varepsilon j = (0,1,2,\ldots n)$; energy only to adopt integer values

and $\Delta\varepsilon j = u_{i+1} - u_i$

and $\Delta U = \varepsilon_{j+1} - \varepsilon_j$; all changes to occur stepwise, one energy unit at a time.

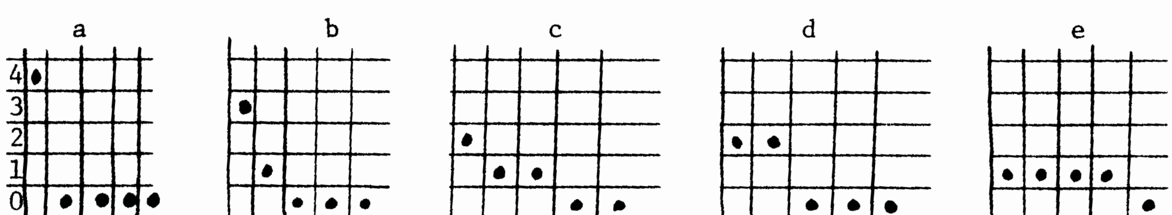
and $p(\Delta\varepsilon_j) = p(\Delta\varepsilon_{j+1})$; all $\Delta\varepsilon_j$

The procedure for generating a distribution is to assign energy units by dipping into a well-shaken urn containing indentifiable discs in equal numbers for all packages, - a "multinomial" urn (Eqn. 5), such that:-

$$p(n_j) = p(n_{j+I}) \text{ ; all } n_j$$

Actually, for such a random process it is only necessary to calculate the <u>a priori</u> probabilities from the permutations and combinations possible at given U and N. For example:-

LET N = 5, and U = 4. We show the possible combinations diagrammatically and the values for permutations $(N!/n_1!n_2!n_3!n_4!n_5!)$ below.



a	b	c	d	e
$(\frac{5!}{1!4!})$	$(\frac{5!}{1!1!3!})$	$(\frac{5!}{1!2!2!})$	$(\frac{5!}{2!3!})$	$(\frac{5!}{4!1!})$
=5	=20	=30	=10	=5

The probability $p(\varepsilon)$ of occupance of a given level depends on the total number of ways a package can be on that level relative to all other possible levels. Writing this simple case out in full we have:-

$$\begin{array}{lccccccccccl}
 & a & & b & & c & & d & & e & & \\
p(o)\propto & (4\times5) & + & (3\times20) & + & (2\times30) & + & (3\times10) & + & (1\times5) & = 175\ ; & p(o) = 0.50 \\
p(\varepsilon)\propto & (o) & + & (1\times20) & + & (2\times30) & + & (o) & + & (4\times5) & = 100\ ; & p(\varepsilon) = 0.29 \\
p(2\varepsilon)\propto & o & + & o & + & (1\times30) & + & (2\times10) & + & o & = 50\ ; & p(2\varepsilon) = 0.14 \\
p(3\varepsilon)\propto & o & + & (1\times20) & + & o & + & o & + & o & = 20\ ; & p(3\varepsilon) = 0.06 \\
p(4\varepsilon)\propto & (1\times5) & + & o & + & o & + & o & + & o & = 5\ ; & p(4\varepsilon) = 0.01 \\
 & & & & & & & & & & \underline{\underline{350}} & \underline{\underline{1.00}}
\end{array}$$

Even here, features which become dominant as N becomes large (and U is close to or greater than N) are clear. The likelihood of all the energy going to one package, $p(4\varepsilon)$ is negligible as is the even distribution in e). The largest number of permutations occurs where packages are most spread out among energy levels (in c), and this combination most closely approximates the overall distribution. With large N and U the latter feature is overwhelming, so that the probabilities of the best-segregated combination $p^*(\varepsilon)$ describes the overall distribution with negligible error. Finally, we note that the probability of occupance falls as ε increases. This will not be true with U appreciably greater than N, when the lowest levels are as unlikely as the highest, - a more realistic representation of many natural energy distributions. However, it illustrates a general principle of the conservatism of energy distribution.

While developments of this kind have received little attention in our field, I suspect that here is one of the few types of theoretical approach that make the discrete-event perspective on magnitude and frequency a worthwhile alternative to the powerful continuous methods which can be adapted from statistical communication theory. Therefore we will pursue the matter a little further, providing some background of use in sampling problems, statistical estimation and extreme value questions.

Given the same initial constraints, the only practical problem with large N is to find the arrrangement where $(N!/n_1!n_2!\ldots n_s!\ldots) = G_a$ is a maximum (G_{max}), since that allows us to find $p^*(\epsilon)$. This is a mathematical problem in which we hold U fixed and observe what happens to G_a with small changes $\delta(n_j)$ $n_1, n_2, \ldots$etc., We can write:-

$$G_a = \frac{\delta G_a}{\delta n_1}\Delta n_1 + \frac{\delta G_a}{\delta n_2}\Delta n_2 + \ldots \frac{\delta G_a}{\delta n_s}\Delta n_s + \ldots$$

$$= \sum_s \frac{\delta G_a}{\delta n_s}\Delta n_s$$

Our deliberations will be more effective if we write:-

$$\Delta(\ln G_a) = \sum_s \frac{\delta(\ln G_a)}{\delta n_s}\Delta n_s$$

noting that:-

$$\ln G_a = \ln \frac{N!}{n_1!n_2!\ldots n_s!}$$

$$= \ln(N!) - \ln(n_1!) - \ln(n_2!) - \ldots \ln(n_s!)$$

$$\therefore \frac{\delta}{\delta n_s}(\ln G_a) = -\frac{\delta}{\delta n_s}\ln(n_s!)$$

With the aid of Stirling,s formula the right-hand term becomes:-

$$\ln(n!) \simeq n \cdot \ln n - n$$

$$\delta[\ln(n!)] = \frac{\delta}{\delta n}[n \cdot \ln(n) - n]$$

$$= n\frac{1}{n} + \ln(n) - 1$$

$$= \ln(n)$$

and

$$\frac{\delta}{\delta n_s}(\ln G_a) = -\ln(n_s!)$$

A solution might now be looked for such that:-

$$-\ln(n_s!) = 0$$

However, this expression does not necessarily satisfy the conditions:-

$$\sum_s \Delta n_s = \Delta(N) = 0$$

and

$$\sum_s \varepsilon_s \Delta n_s = \Delta(U) = 0$$

However, we can take account of these two constraints by multiplying by, as yet, undetermined constants or Lagrangian multipliers L_1, and L_2 (see Sokolnikoff and Redheffer op. cit. p.342). Since adding zero does not affect our original development we can now write:-

$$\Delta(\ln G_a) = \sum_s \frac{\delta(\ln G_a)}{\delta n_s} \Delta n_s + L_1 \,_s \Delta n_s + L_2 \,_s \varepsilon_s \Delta n_s$$

$$= \frac{\delta}{\delta n_s}(\ln G_a) + L_1 + L_2 \varepsilon_s$$

$$= -\ln(n_s) + L_1 + L_2 \varepsilon_s \quad ;\text{giving stationary } \ln G_{max}$$

$$= 0$$

Thus, the most probable values n_s^* of the n_s are found from:-

$$\ln n_s = L_1 + L_2 \varepsilon_s \tag{15}$$

and

$$n_s^* = \exp[L_1 + L_2 \varepsilon_s]$$
$$= \exp(L_1)\exp(L_2 \varepsilon_s)$$

Thus the probability of occupying an energy level is an exponential function of energy. Since overall energy will be normally distributed one can see some analogy here with normally distributed event-counts that have exponential recurrence intervals. Meanwhile, the above development also defines a measure of the uncertainty, or "statistical entropy" which depends on the same constraints, but is given by $\ln G_{max}$, so that it varies more slowly and at a decreasing rate with N and U (see Fast, 1962 for a discussion and also more advanced treatment of above material). The well-known result in statistical thermodynamics that entropy tends to a maximum, is no more than a formalisation of arguments above, that the state having the greatest number of micro-states is also the most probable state. As a corollary, statistical entropy also gives a measure of our inability to distinguish between configurations of micro-states.

If it is found that the magnitude-frequency distribution of a set of natural events or a component of their measurements obeys a normal distribution, then some random dispersal of energy, mass etc., along the lines depicted above might be looked for. Just such reasoning as this has been applied to sea waves in which height and slope are approximately Gaussian variables (Cartwright, op. cit.).

We are not confined to the normally distributed case. Obviously, by modifying the constraints, the above development is open to a variety of other results (i.e. with biased sampling, irregular energy levels, non-regular dispersion etc.,). At this point we need to note, also that different magnitude variables of the same event often have different distributions. The distribution of stage heights on a river is quantitatively unlike the associated distribution of discharge.

In deriving a distribution for wind speeds, closely approximating actual measurements, Davenport (1968) used essentially similar reasoning to that above. Beginning with the assumption of an isotropic wind with randomly distributed, horizontal directional components, he arrives at a Rayleigh distribution for the speeds over a small range of magnitude and in a small directional segment. This is, as we would expect, an exponential distribution of the form:-

$$p(x) = e^{-\frac{x^2}{2\sigma^2}} \quad ; \; x>0 \qquad (16)$$

RAYLEIGH DISTRIBUTION

Incidentally, this particular result illustrates how the choice of measurement or variable influences the distribution obtained. A Rayleigh probability function obtains when we study the envelop swept out by a rotating-vector following a Gaussian wave-form. The probability density function of this envelop cannot itself be Normal (Stewart, 1960, Sec. 11, 5.). It is not surprising that the Rayleigh describes the distribution of total sea-wave heights under parallel constraints to those used by Davenport (Cartwright op. cit. p.582). Davenport (op. cit.) also proceeded to show how the distribution can be modified to account for anistropy and a "prevailing wind" situation. He arrives at a Weibull distribution:

$$p(>x) = e^{-\left(\frac{x}{c}\right)^{k}} \tag{17}$$

Basically this just allows some flexibility in the exponent, although his empirical results indicate that the exponents are normally so close to 2 that the Rayleigh would be a very close fit.

There is an alternative theoretical approach which might lead to such distributions. If the magnitude of some event relates more-or-less directly to a physical law or experimental regularity of a particular form, the event-distribution may reflect that form. The rate of nucleation of ice has been shown experimentally to increase exponentially with increased supercooling and to decrease exponentially for fixed temperatures (Vali and Stansbury, 1965). The size distributions of hail-stones and magnitudes of hail-shafts, might be expected to reflect these regularities. Provisional evidence indicates they do though not in a simple fashion (ibid. p. 25). Likewise, the changing thermal conductivity of ice as it thickens is an exponential decay function, and we might expect the thicknesses of sea- or lake-ice encountered by ice-breakers to be related to this. In each case, of course, the individual events or measurements may be random variables in similar ways to our energy-level model above. However, it should be apparent that some kind of stochastic "growth" or "decay" process would be more directly relevant here.

A distribution not mentioned so far which occurs frequently in natural situations is the lognormal. We have considered exponential functions of the form $x = e^{y/a}$. Their inverse is a logarithmic function of the form y = a ln x. If plotted on a line, the actual measurements for such a function would increase geometrically;but their logarithms, arithmetically. In particular, when plotted as a frequency function on arithmetic paper the logarithms of the observations should plot as a straight-line. This property is associated with normality, and leads to the idea a "lognormally" distributed phenomena. The probability density function used reflects the relation to the Normal (Eqn. 7), and is:-

$$p(x) = \frac{1}{(x-c)\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(x-c)-m}{\sigma}\right]^2} \quad ; \ x > c \qquad (18)$$

$$= 0 \quad ; \ x \leq c$$

Evidently, ln (x-c) is normally distributed.

Just as the Normal distribution was seen as developing from the sum of many independent, identically distributed random variables; the Lognormal would arise from the products of such variables. The kind of situation where the Lognormal fits recognisable physical regularities is in certain "growth" properties. Viewed over time, many of the "events" we are interested in grow through accumulated mass, energy, or through concentration or spreading. This may occur by simple addition, but, as in many biological and economic phenomena, - where the Lognormal has played a large role, - growth or intensification of natural "packages" may also depend upon the existing size. Following Cramer (1945, p. 219-20), suppose our "package," - say, some aspect of a storm, warm-spell, flood wave - grows by a series of random inputs $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_s$, and let x_u be the magnitude of event produced by the inputs $\varepsilon_1, \ldots, \varepsilon_u$. The increase due to input ε_{u+1} is then proportional to ε_{u+1}

and some function $g(x_u)$ of the size of the event:-

$$x_{u+1} = x_u + \varepsilon_{u+1} g(x_u)$$

We then have

$$\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_s = \sum_s \frac{x_{u-1} x_u}{g(x_u)}$$

If each input gives but a small contribution we can write:-

$$\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_s = \int_{xo}^{x} \frac{d(t)}{g(t)} \tag{19}$$

Cramer shows that if $g(t) = t$ then log x is asymptotically normally distributed (see Aitchison and Brown 1957, Ch. 3 for more elaborate discussions). This so-called "law of proportionate effect" has some severe logical and theoretical limitations but the associated distributions have proved very useful. Again, a number of physical laws relevant to natural events or their impact on man, are essentially lognormal functions. The relations between flow velocity and competence of particle transport by water follow Stokes' law for fines, and the "impact law" for coarse materials. Both laws are lognormal functions under logarithmic transformation. Tolerance of stress, and sensory "adaptation" in certain animals and man seem to follow a logarithmic law (see also Gaddum, 1945 and related correspondence). Erosional events, or human response to certain environmental conditions may be related to such regularities.

More generally, it is possible to represent various distributions as Normal and to associate parameters with physical effects, using the Edgeworth-Kapteyn method. Let a distribution be described by:-

$$p(x) = \frac{1}{\sigma_g \sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{g(x)-m}{\sigma g}\right]^2} \tag{20}$$

where $g(x)$ is chosen so as to be normally distributed, and m and σ_g can be estimated. Any random variable x which fits this distribution may be regarded as the limit of a sequence of random variables of the form $t_1 = t_o + z_1 h(t_o)$, $t_z = t_1 + z_2 h(t_1) \ldots$ with z_1, z_2 ..., normally distributed random variables and $t_1 + t_2 + \ldots = \int_{x_0}^{x} \frac{d(t)}{h(t)} = g(x)$. The z_1, z_2,... can then be thought of as the response magnitudes of the physical process which successively produces t_1, t_2, ... The analogy with Cramer's development above is obvious (c.f. again 2.3. pp.   ).

2.5. Extreme Event Probabilities

Extreme events are those whose magnitude is very high or low, and in most cases means they are also rare events. Geographers may, of course, be interested in events that are extreme relative to those of other regions, though common in an extreme environment, but I know of no work on probabilities of "spatial extremes."

Probabilistic study of extreme events has been closely tied to the mathematical theory of extreme values, and has dealt almost entirely with random, stationary series. Most substantive studies operate on the basis that periodicity, trend or serial correlation can be removed or ignored. The mathematical theory of extreme value probabilities is associated particularly with the work of von Mises, Frechet, Fisher and Tippet, Gnedenko and E. J. Gumbel. Undoubtedly, Gumbel's work has been the most detailed and exhaustive, and the most widely applied. The present discussion will be based mainly on Gumbel (1954a; 1954b; 1958a; and 1958b).

One of the most vexing questions in extreme value statistics is the sense in which it is impossible to win by honest means. There are two aspects to this; first, statistical estimation of parameters of a distribution requires a sufficiently large and representative sample. Second, the sample must be statistically homogeneous. For extreme events these two requirements are in some degree self-defeating. Given enough time even the most conservative properties of the earth or universe change. Without a longish period of time we cannot obtain an adequate sample. The same reasoning applies if we cast our net widely in a spatial sense.

There are two possibilities. If we can discover the overall or initial distribution in which the extreme values arise, then we can determine their probabilities directly from the structure of the distribution. Generally, investigators have found they cannot define the initial distribution. We then consider the ways extreme values can behave whatever the initial distribution; the "distribution free" approach. The two approaches are not mutually exclusive. Extreme value behaviour of known distributions must fit one or other of the models developed by the second approach.

The core of Gumbel's reasoning, based mainly on Fisher and Tippet (1928) is to evaluate in mathematical terms, how a distribution can behave as sample size tends towards infinity, and then define criteria for deciding whether a given set of extreme values behaves as one or other of the possible limiting or asympototic distributions of the largest value. The decision criteria have been closely associated with the development of special graph paper and plotting position conventions. These have done more to give Gumbel's work wide circulation - and misinterpretation(!) - than anything else, but will not be dealt with here. We look at the probabilistic reasoning only.

In his book, Gumbel (1958a) devotes some space to the extreme value distributions of probability models such as we have already discussed. Our examination of the occupation of energy levels would lead us to expect that exponential functions play a central role. For an Exponential distribution of the form $f(x) = e^{x}$, Gumbel shows (1958a, pp. 113-4) that the most probable or modal largest value is given by:-

$$\bar{x}_n = \ln n \qquad (21)$$

which relates, of course, to our earlier discussions. He further shows that the mode is here equal to the important statistic characteristic largest value, u_n, found from:-

$$F(u_n) = 1 - 1/n \quad ; \; n \geq 2 \qquad (22)$$

In general, an Exponential distribution, and also the Normal, Lognormal and Gamma distributions of the types defined above, will have asymptotic largest value probabilities of the same type (Type I below). But, as we might expect, all the asymptotes are of exponential form.

Where the initial distribution is unknown, and insofar as the data can be treated statistically, Gumbel showed that the asymptotes can only adopt three forms, as determined by the following conditions:-

1) TYPE I "The Exponential": a probability function which is unlimited to the right but converges towards unity in about the same way as e^{-x} approaches zero. Such distributions possess all moments.

ii) TYPE II: "The Cauchy or Pareto Type" in which the initial distribution is also unbounded to the right but has a longer tail such that all moments do not exist. This distribution was developed by Fréchet.

iii) TYPE III: "The Limited Type" in which the initial distribution bounded to the right and the probability reaches unity at some limited value.

For these conditions Gumbel gives the following asymptotic probabilities $\phi^{(m)}x$, of the largest values:-

i) $\phi^{(1)}(x) = \exp\left[-e^{-\alpha(x-u)}\right] \quad ; -\infty<x<\infty$ (23)

ii) $\phi^{(2)}(x) = \exp\left[-\left(\frac{u-\theta}{x-\theta}\right)\right]^{k} \quad ; x>\theta ; u>\theta$ (24)

iii) $\phi^{(3)}(x) = \exp\left[-\left(\frac{w-x}{w-u}\right)^{k}\right] \quad ; x<\omega ; u>\omega$ (25)

with, α, a parameter of the dimension $1/x$;u, a parameter of the dimension of x called the characteristic value (Eqn above); θ the lower limit and ω the upper limit of a variable. Since the conditions for the smallest values are symmetrical, extreme low value probability distributions can be obtained by appropriate reversal of terms. The three types can be distinguished graphically by virtue of the fact that their second differentials or curvatures differ:-

$$\frac{d^2x}{dy2} \begin{cases} = 0; \ \phi^{(1)}x \\ > 0; \ \phi^{(2)}x \\ < 0; \ \phi^{(3)}x \end{cases} \qquad (26)$$

(see Jenkinson, 1955)

These three asymptote's of Fisher and Tippet which Gumbel elaborates, play a dominant role in the study of natural extremes, having been used for floods, high winds, droughts, waves, snow-fall, earthquakes, rainfall and maximum boulder-size in sediments (see Table II at the end). Short reviews and modifying statements are numerous (e.g. Kendall, 1959; Benson, 1967; Gumbel, 1967; Jennings and Benson, 1969; also Langbein 1949; Dalrymple 1960; Thom, 1968).

Gumbel's essentially theoretical approach to the problem (i.e. 1968a, p. 345) seems close to the theme of the present paper. Unfortunately, from a physical point of view the extreme value work to date is severely limited in its usefulness. Gumbel's theoretical approach is in mathematical formulation and fails to penetrate his application to material situations. The principle strengths of his methods are also their fundamental weaknesses. The fact that the asymptotic stability assumption allows us to describe extreme values without knowing their initial distribution is useful, but it also means that the method can tell us little about the underlying process. Alternatively, if we can define the latter we know all we need to know to determine the extremes. The extreme value interpretation is especially useful in allowing us to use the highest (or lower) measurements of an observation series. Commonly, these are annual maxima or minima. Nearly all studies available use such observations. But these are extraordinarily crude and insensitive sorts of observation, and I am aware of no adequate definitions of their physical meaning.

To illustrate, I have used the Gumbel methods to study extreme discharges of the Upper Indus. In the hundred year, once-daily record it is possible to identify they type of flood condition for many of the annual peaks. These range from extended high flow periods due to snow and ice melt, to short flood waves either from the same source or, from glacier dam-bursts, landslide dam-bursts or severe rain storms. All the evidence is that these differing flood conditions have different frequency distributions. Using the Gumbel method the net result is a Type I asymptotic distribution of the annual maxima; apart from some vexing historical floods. It can only be concluded that the distribution has some emprical justification but this may be as much a measure of its insensitivity as its versatility; (see also Moran, 1957, and Yevdevitch in Discussion of Gumbel 1967, p. 173).

In fact, there is hardly any work on extreme events in the sense of well-defined, bounded physical states, except in the terms discussed in the next section.

2.6 Impossible and Improbable Events

In scientific speculation, - the search for theory, - a valuable working hypothesis may be that whatever does not defy physical laws will happen sooner or later. In a deterministic framework this idea of "natural fulfilment" adds little information, requiring the all-knowing intelligence described by Laplace. In probabilistic terms the idea has useful and meaningful formulations, that can be used even without the information to apply extreme-value theory. Thus, while the concentration of energy in one or two "packages" of the occupancy analogue is highly unlikely it is not impossible: indeed if it could not occur in the long run it would have to have probability zero. Very exceptional events of this type are of interest because they can have exceptionally high magnitudes or crucial configurations whose impact extends far into the future, perhaps with irreversible consequences.

Obvious examples are genetic mutations, natural or man-made calamities, or invasions of exotic flora and fauna (see Gretener, 1967). Here is a meaningful field for geographical "exceptionalism"! Often, the significance of improbable but not impossible events may not be in explaining nature so much as showing the weakness of other theories. Many of the great battles over culture contacts or innovations in human societies evaporate in the face of long-run probabilities. Given enough time, the improbable becomes certain.

The basic problem is, of course, arriving at some estimate of probabilities. However, in these cases we are usually dealing with time or space dimensions which make ordinary criteria of accuracy irrelevant: simply obtaining orders of magnitude is highly instructive. For the latter, circumstantial evidence may allow one to set the all-important inner or outer limits. Another aid here which has greatly helped some sciences, is what the astronomer Carl Sagan calls "the assumption of mediocrity." It has some affinities with the Uniformitarian principle but has only the status of a working hypothesis not a necessary principle of investigation. We simply assume that what is unknown is like the known until we have evidence that it is not. For instance, without better knowledge, we would use short-run spatial averages to estimate time-frequencies at a place, or time frequencies to obtain spatial probabilities for unexplored areas, and examine the consequences. Physical geographers have had bad experience with verbal and visual relating of contemporary features according to evolutionary stages, but that was because there were no immmediately testable, refutable consequences built into the theory. Astronomers, from a relatively much shorter observation period have developed rigorous and elegant theories of stellar evolution using spatial sampling (e.g. Schwarzschild, 1958). Hydrologists are developing useful methods of estimating hydrologic characteristics of ungauged rivers by extrapolation from gauged ones. In the same way we may be able to deduce order-of-magnitude probabilities for important extreme events without local observations.

Once we can arrive at some idea of the range of probable recurrence, the mathematical formalities are fairly straightforward. Since we generally have no basis for finding deterministic components we assume that the exceptional event is random, and the probabilities turn upon the kinds of questions used to introduce the Geometric and Negative Binomial distributions (Eqns. 9 and 10). Since these are rare, singular events, the Poisson distribution is the obvious basis for calculation. The probability of recording at least one "success" after exactly x trials with a Poisson-type urn process is:-

$$p(k = 1;x) = 1 - e^{-xp} \qquad (27)$$

while the probability of k = n successes is found from the general expression for the distribution (Eqn.8). Using these assumptions, Gretener (1967) tabulated some representative values for given annual probabilities (p) over various periods of years. For instance, if $p = 10^{-2}$, on average there will be one event in 10^2 years, ten events in 10^3, and 100 events in 10^4 years.

Let us illustrate the point by some simple calculations. Major natural disasters are still treated as "Acts of God" by insurance companies and most laymen. We only expect them to occur in our home area after many generations. However, assume that in the ordinary run of events, any 1000 km^2 area has an average chance of a major natural upset once in 1000 years,- p = 0.001 or 10^{-3} for any given year. Only about 0.25 of the earth's land area has sufficiently dense or rich populations to register globally significant damage. The sums are:-

Number of richly populated units = $(3.5 \times 10^7)\ (10^{-3}) = 3.5 \times 10^4$

Hypothetical rate of major disasters = $(3.5 \times 10^4)\ (10^{-3})\ yr^{-1}$
= 35 per year

This just happens to be very close to the average for a canvass of reported multi-million dollar or high-death natural disasters in the last 20 years. If they are indeed "Acts of God," He must be pretty mad at somebody most of the time. At least, we can see that local awareness is quite inadequate to evaluate the real dimensions of the extreme natural hazard problem, which, far from involving the unusual is, in a global sense, an ever-present component of man's environment. We also see how, even at this crude level, the use of spatial averages and the "mediocrity" idea given a new dimension to an old problem which is still being investigated using "exceptionalist" questions.

PART III Logical Design in the Study of Discrete Events

3.1 The Present Situation

Much of the work on discrete natural events, even when mathematically sophisticated, is at best exploratory, at worst misleading. The reasons behind this rather drastic statement are two-fold. First, in the absence of clear, physical frameworks, the statistical analysis which examines and reflects the largest range of relevant observations in the most varied ways, is least suspect. Much discrete event work not merely grossly simplifies the data before analysis (as we did in our lake-level example), but only checks the fit of one or two related distributions. Second, as a science moves from verbal, typological identification of objects and variables, to a quantitative, parametric approach, the pattern of significant continuities and discontinuities may change radically. In terms of measured stress, energy, areal extent, duration, generative process, controlling conditions and so on, phenomena which everyday experience suggests are discrete and special merge or are otherwise closely related to apparently distinct phenomena or conditions, and the latter may help provide the main clues in interpretation. Conversely, significant differences appear in phenomena thought to be the same. These points are especially relevant when we consider the probability distribution underlying a set of events. Methods which fail to allow for these possibilities, or are not based in well-articulated, testable physical theory will be inferior and probably misleading. Since most examples in Table I and II only look at the experiental-event record, often interpreted as a point measure of doubtful physical meaning, their significance is very ambiguous. (Important exceptions are Davenport, 1968, or Pierson 19). We need, therefore, to look very carefully at the logic of discrete event studies if they are to be anything but exploratory devices.

3.2 Logical Components

Any statistical investigation of natural phenomena must be formulated in terms of at least three components which we will call:-

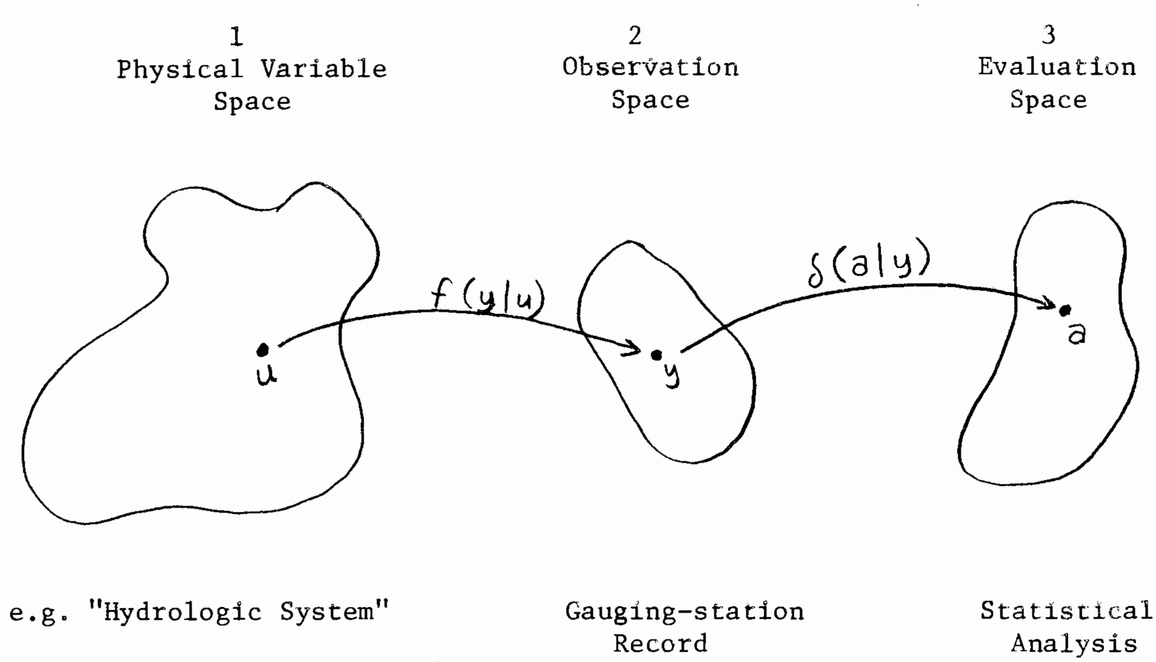
i) The physical variable space.

ii) The observation space.

and iii) The evaluation space.

Essentially, studies of nature attempt to "map" each of these into the other rather as is shown by the diagram (Fig. 1).

FIGURE 1.



Our objective is to show the logical status and equivalence relations which can hold between these components.

3.2.1. The Substantive Area: Physical Variable and Observation Space

Scientists tend to regard their subject-matter as being objectively present in a publicly observable environment. Although some empiricists and positivists object, this is a fundamental logical position of science.* Nevertheless, such environments contain infinitely many elements, some of whose measurable characteristics appear to vary continuously, some discretely, or both on different occasions. In general, there exist infinitely many observations or "messages." In an investigation we define the meaningful source of information or observation space. Logically, the situation is identical to that of a communications device for which a "message" put out by a source is only meaningful to the extent that it can be read and correctly interpreted by the receiver. Though it has logical status, we cannot know the physical variable space directly, but only through the defined observation space. We possibly never know the shape of the former, although, through the procedure of testing scientific theories in terms of crucial consequences we give that space high status. But then, the output of natural processes is only relevant to us to the extent that we have means to read and interpret that output in our own "languages." Hence, while communication engineering is operationally different in that the properties and goals of its sources are usually known and in humanoid languages, logically the position of the receiver and its output is identical to that of the observer or observing instrument and their output, in a scientific investigation.

In discrete event studies, the simplest observing convention is to treat the world in terms of one event-type. Here, the physical variable, or source, can only give two message, $u_1(t)$ indicating that the phenomenon

*It is the position recognised by Russell and Whithead in the Principia is present, and $u_o(t)$ when absent, and the observations are essentially a string of noughts and ones. But what are we doing in the mapping of "real-world" into observation space when, say, the event is a tornado or a male animal? Actually, identification of the message $u_1(t)$ requires the simultaneous verification of very many propositions, as indeed may $u_o(t)$. When we come to map the observations into the evaluation space, the identification criteria will have no status. Even the theory of observational errors cannot be used with non-specific data. Nevertheless, the identification criteria do help determine the probability distribution by defining the information which can be used in analysis.

Since it is theoretically possible to organise and select observations of any kind so as to fit any distribution, one would want a formal statement of how observations sampling the real world should have a high probability of reflecting that world. Often, it is suggested that the only issue here is obtaining a sufficiently unbiassed and large (i.e. "representative") sample. In fact, the key scientific issue is whether the event space defined for the variable matches the probability space of the source. The latter can only be searched out by the successive testing of hypotheses couched in the terms of physical theory and its consequences.

3.2.2 The Interpretive Area: The Evaluation and Observation Spaces

At the opposite end from the "real-world" phenomenon is in our case the probabilistic viewpoint involving a system of logic for the assigning of probabilities to sets of numbers or other symbols, and formal tests of the "fit" of these to probability distributions.

Notice, however, the pivotal position of the observations. They not only overlap the logical position of the physical variable, but also the evaluative area. In the latter case there are two aspects. First, there must be a formal statement and defense of the type and arrangement of observations that permits them to be evaluated by the probability calculus. The observations must conform to the logic of statistical method. Second, it is generally the case that rational, often statistical, criteria are built-into the observing process so that evaluation or "decision-making" about what is being observed may precede the output of data as well as be used to evaluate that data.

Once we have a set of acceptable observations the evaluation space can involve at least three processes; simplification sampling or transformation of the basic data, fitting of a priori distributions, and tests to decide upon the most appropriate of the latter. In each of these cases, the actual procedure is determined by statistical (or deterministic) decision-making. From the physical aspect, it is the rationale of this decision-making which is central. The decisions must be logical and suited to probabilistic designs, but above all they must reflect overt models or hypotheses about the formal or behavioural properties of the substantive phenomenon. Anything else is "talking about talking," and removes the possibility of meaningful tests in the face of the facts.

3.3 The Logical Status of Discrete Events

The above statements apply throughout statistical work in science. They are raised here because their import rarely penetrates work on discrete natural events. To ignore the obvious is worse than to state it.

We define the mapping of physical event into an observation space by our "criteria of recognition" (1. above). For the case of discrete natural events I want particularly to point to the "logical jump" which can occur here. This arises especially when we have some central phenomenon of interest (e.g. flood damage, drought, effective rainfall), while the data are standard at-a-station records, - the most common situation.

Take as an example the probability of occurrence, duration and degree of flood-damage at a place, where guaging-station records are to be used. Our substantive verbalised interest directs us only to stage readings above some "no-damage" threshold. Now, as "input" to social action, only these readings are relevant. Are they, however, a physically separate set with a separate generative process, in a hydrological sense? Almost certainly not. Is it possible that our undifferentiated flood events have more than one source, and belong to more than one probability distribution? Certainly it is possible. Flood readings are essentially continuous with no-flood readings. If our object is anything more than order-of-magnitude estimates for forecasting, these points must be considered. (Usually they are not). For what we are in fact doing is taking an <u>event,</u> in the sense of this paper, and defining it, by a certain equivalence relation, as a statistical event: namely, the type of a set or class of points in a probability space such as might be represented by a Euler diagram. In Fundamental research we need very good grounds for truncating, warping or otherwise altering, or "sampling" in, a probability space. Having chosen the station records as our input to statistical analysis, they form the basis of our statistical event definition and the means of defining the probability space(s). The flood-events form a sub-class within the data and may belong to one or more statistical event types. In cases such as this, the physical meaning of discrete readings referred to earlier, is also a major issue.

In general, since we often lack sound physical theory on which to base equivalence-relations of real-world and observations, the answer is to look at as much of the data as possible, with as sensitive an analysis as possible; preferably one which allows the data to "speak for itself" to a large degree. To this extent, developments outside of those discussed in the present paper seem more appropriate; methods such as spectral analysis, "self-similar snythetic hydrology" (Wallis and Mandelbrot, 1968) and other outgrowths of statistical communication theory. These provide powerful and sensitive computational procedures that allow one to scan large volumes of data and analyse events over wide frequency ranges with considerable rigour. They need mention here because so much of the work on discrete natural events, in fact uses observations such as once-daily station readings which are susceptible to these forms of analysis. As such, problems of determining the initial distributions for extreme or rare events may be best solved by them. Of course, the logical problems described above are still large issues whatever analytical method is used. Nevertheless, there is still a huge area in which discrete event probabilities of the kind we have described is relevant and valuable. Work of this kinds is also an important complement to other analyses. However, the literature indicates that the exploratory phase of trying to see which distributions fit experientally-defined objects, can no longer make much contribution. We now need to explore and define the ways in which the distributions can arise under various constraints, and the logic of proposed congruence of natural and probabilistic events.

TABLE 1

EXAMPLES OF DISCRETE NATURAL PHENOMENA ANALYSED IN TERMS OF VARIOUS PROBABILITY MODELS

BINOMIAL DISTRIBUTION		
a)	Low precipitation: annual frequency of dry months at Oxford, U.K., 1851-1943	Brooks and Carruthers (1953, p. 71)
b)	Frosts: frequency of frost-days per month (April) at Greenwich, U. K., 1841-1905	Ibid (p. 72)
POISSON DISTRIBUTION		
a)	Hail: mean annual frequency at a station ($\lambda < 2$)	Thom (1957)
b)	Precipitation: counts of nuclei in small parcels of air	Scrase (1935)
c)	Heavy rainstorms: London, U. K., 1871-1931	Brooks and Carruthers (op. cit., p. 79)
d)	Meteorite strikes on (potential) human targets	LaPaz (1958, p. 229)
NEGATIVE BINOMIAL		
a)	Hail: mean annual frequency at a station ($\lambda > 2$)	Thom (1957)
b)	Tornadoes: frequency	Thom (1963)
MODIFIED POISSON (Eggenberger and Polya's)		
a)	Rain: monthly rainfall days	Wanner, E. (1942)
b)	Snowfalls: Switzerland, 1901-1940	Uttinger (1945)
OTHER WEAKLY "CONTAGIOUS" CASES		
a)	Earthquakes: frequency of aftershocks	See bibliography in Kitagawa (1965)
NORMAL		
a)	Sea-waves: wave height, up-wind and cross-wind wave slopes (n = 2000)	Cartwright (1962)
b)	Sea-waves	Pierson (1954) Putz (1954)
c)	Rainfall: cube roots of i) daily amounts at Jakarta and Zurich and ii) monthly rainfall at Halifax, Nova Scotia	Stidd (1953) Bruce and Clark (1966)

Table 1 (Cont'd)

d) Temperature: frequency of discrete readings, Scilly, U. K., Aprils, 1928-1937	Brooks and Carruthers (op.cit., p. 98)
GAMMA	
a) Sea-waves: height	Longuet-Higgins (1952)
b) River levels: recurrence of exceedances (various levels)	McGilchrist et al. (1969)
c) Damaging hail storms: probabilities of regional occurrence in Iowa	Decker (1952)
d) Precipitation: drought occurrence ("incomplete" Gamma model)	Barger and Thom (1949)
MARKOV PROCESS	
a) Temperature: cold spells at a place	Caskey (1964)
b) River level: recurrence of exceedances (various levels)	McGilchrist et al. (1968 and 1969)
c) Precipitation: wet and dry days in rainy season, Tel Aviv	Gabriel and Neumann (1962)
d) Surface dew point, Minneapolis, July	Gringorten (1968)
EXPONENTIAL	
a) River levels: exceedances (various levels)	McGilchrist et al. (1969)
b) Wind speed: frequent range (actually Rayleigh and Wiebull)	Davenport (1968)
c) Wave heights: trough-to-crest (Rayleigh)	Longuet-Higgins Cartwright (1962)
d) Solar Radiation: storm-bursts of Spectral Type I	Kundu (1965, p. 189)
LOG-LOG	
a) Meteorites: cumulative size-frequency distribution of meteoric masses and meteorite craters	Shoemaker (1966) Hawkins et al. (1958, pp. 730-731)
LOGNORMAL	
a) Tsunamis: heights of "runup" (R) or shore inundation (% frequency against $R/\overline{R}$)	Van Dorn (1965)
b) Hydrologic series: various examples	Ven Te Chow (1954)

Table 1 (Cont'd)

c)	Streamflow: Maxima of average daily flows, Elbow R., Alberta, 1908-1964	Keeping (1967)
d)	Tornadoes: dimensions of damage swath	Thom (1963)
e)	Flood damage magnitude: U. S. A.	American Insurance Association (1956)
f)	Earthquakes: magnitude and frequency	Asada (1957)

TABLE II

EXTREME EVENT PROBABILITIES

GUMBEL (FISHER-TIPPET) TYPE I		
a)	High River Flow (Floods) (e.g. in U.S.A., Brazil, New Zealand, France, Canada, China, etc.)	Gumbel (1954a, 1954b, 1958a, 1958b) Jenkinson (1955) Boyd and Kendall (1956) Kaczmarek (1957)
b)	Low River Flow (Droughts)	Gumbel (1954a, 1958a)
c)	High Winds	Court (1953) Davenport (1958)
d)	Maximum Vapour Pressure	Jenkinson (1955)
GUMBEL (FISHER-TIPPET) TYPE II		
a)	High River Flows	Gumbel (1954a, 1958a)
b)	Low River Flows	Gumbel (1958a)
c)	High Winds	Thom (1968)
GUMBEL (FISHER-TIPPET) TYPE III		
a)	High River Flows	Jenkinson (1955)
b)	Low River Flows	Gumbel (1958)
LOGNORMAL		
a)	High River Flows	Olofgors (1951) Ven Te Chow (1955) Kaczmarek (1957)
LOG-PEARSON TYPE III		
a)	High River Flows	Water Resources Council (1967) Jennings and Benson (1969)
PEARSON TYPE III		
a)	High River Flows	Foster (1924) Kaczmarek (1957)

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